

Uncovering latent singularities from multifractal scaling laws in mixed asymptotic regime. Application to turbulence

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In this paper we revisit an idea originally proposed by Mandelbrot about the possibility to observe “negative dimensions” in random multifractals. For that purpose, we define a new way to study scaling where the observation scale τ and the total sample length L are respectively going to zero and to infinity. This “mixed” asymptotic regime is parametrized by an exponent χ that corresponds to Mandelbrot “supersampling exponent”. In order to study the scaling exponents in the mixed regime, we use a formalism introduced in the context of the physics of disordered systems relying upon traveling wave solutions of some non-linear iteration equation. Within our approach, we show that for random multiplicative cascade models, the parameter χ can be interpreted as a negative dimension and, as anticipated by Mandelbrot, allows one to uncover the “hidden” negative part of the singularity spectrum, corresponding to “latent” singularities. We illustrate our purpose on synthetic cascade models. When applied to turbulence data, this formalism allows us to distinguish two popular phenomenological models of dissipation intermittency: We show that the mixed scaling exponents agree with a log-normal model and not with log-Poisson statistics.

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Multifractal processes are random functions (or measures) that possess non trivial scaling properties. They are widely used models in many areas of applied and fundamental fields. Well known examples are turbulence, internet traffic, rainfall distributions or finance. For the sake of simplicity we will consider only non-decreasing multifractal processes (which increments define a multifractal measure) denoted hereafter $M(t)$. In the sequel $M(I)$ will stand for the measure of the interval I , $M(I) = \int_I dM$ and $M(t, \ell) = M([t, t + \ell])$. Multifractals are characterized by the scaling of the partition functions: If one covers the overall sample interval of length L with L/ℓ disjoint intervals of size ℓ , $\{I_\ell(i)\}_{i=1 \dots L/\ell}$, one usually defines the order q partition function which scaling behavior defines the exponent $\tau_0(q)$:

$$Z(q, \ell) = \sum_{i=1}^{L/\ell} M[I_\ell(i)]^q \underset{\ell \rightarrow 0}{\sim} \ell^{\tau_0(q)} \quad (1)$$

where the limit $\ell \rightarrow 0$ simply means that $\ell/T \rightarrow 0$, T being the correlation scale usually referred to as the integral scale. When $\tau_0(q)$ is a (concave) nonlinear function of q , the measure $M(t)$ is said to be multifractal or intermittent. Within the multifractal formalism introduced by Parisi and Frisch (see e.g. [1]), the non-linearity of $\tau_0(q)$ is interpreted in terms of fluctuations of

pointwise singularity exponents of the measure. Indeed, according to this formalism, $\tau_0(q)$ is obtained as the Legendre transform of the singularity spectrum $f_0(\alpha)$ that gives the (Hausdorff) dimension of the sets of points t of singularity α ($M(t, \ell) \sim \ell^\alpha$). Therefore q can be interpreted as a value of the derivative of $f_0(\alpha)$ and conversely α is a value of the derivative of $\tau_0(q)$. Multifractality is also closely related to the notion of stochastic self-similarity: The measure $M(t)$ is self-similar in a stochastic sense if, for all $s < 1$, $M(st) =_{law} s W_s M(t)$ where W_s is a positive random weight independent of M . It can be easily shown that this stochastic equality implies that the expected value of $Z(q, \ell)$ (i.e., the order q moment of the measure) behaves as a power law, i.e., $\mathbb{E}[Z(q, \ell)] = C_q \ell^{\tau(q)}$ where $\tau(q)$ is nothing but the cumulant generating function of $\ln W_s$. If one uses the terminology introduced in physics of disordered systems, the exponent $\tau(q) \sim \ln \mathbb{E}[Z(q, \ell)]$ is defined from an “annealed” averaging while $\tau_0(q) \sim \mathbb{E}[\ln Z(q, \ell)]$ is the analog of a free energy computed as a “quenched” average. As it will be discussed below, these two functions can be different for large values of q .

The paradigm of self-similar measures are random multiplicative cascades (for the sake of simplicity we will exclusively focus, in this paper, on discrete cascades but all our results can be easily extended to recent continuous cascade constructions [2, 3, 4]) originally introduced by the russian school for modelling the energy cascade in fully developed turbulence and to which a lot of mathematical studies have been devoted [5, 6, 7, 8]. Let us

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summary the main properties of these constructions. The integral scale from which the cascading process “starts” is denoted as T . A dyadic discrete cascade is build as follows: A measure is uniformly spread over the starting interval $[0, T]$ and one splits this interval in two equal parts: On each part, the density is multiplied by (positive) i.i.d. random factors W such that $\mathbb{E}[W] = 1$. Each of the two sub-intervals is again cut in two equal parts and the process is repeated infinitely. It is convenient to introduce the random variable $\omega = \ln(W)$ which probability density function will be denoted as $g(x)$. Thus $g(x)$ is Gaussian or Poisson for respectively log-Normal and log-Poisson cascades. The stochastic self-similarity property of the limit measure M associated with previous iterative construction can be directly proven and it is easy to show that $\tau(q)$ is related to the cumulant generating function of ω :

$$\tau(q) = q - \ln_2(\mathbb{E}[e^{q\omega}]) - 1 \quad (2)$$

Moreover, as first established by Molchan [7, 9], the multifractal formalism holds for random cascades and one has the following relationship between $\tau_0(q)$ and $\tau(q)$, for $q > 0$, (note that a similar relationship holds for negative values of q [7]):

$$\tau_0(q) = \tau(q) \text{ if } q \leq q_0 \text{ and } \tau_0(q) = \alpha_0 q \text{ if } q > q_0 \quad (3)$$

where q_0 is the value of q corresponding to minimum value of $f_0(\alpha)$ (in general $f_0(\alpha_0) = 0$). This discrepancy between the annealed and quenched spectra has been extensively studied on a numerical ground in ref. [10] and was referred to as the “linearization effect”. It has notably been observed to be independent of the nature of the cascade and of the overall length L of the sample. As it will be emphasized below this effect has not been properly taken into account in the literature of turbulence (see e.g. [11]).

One of the goals of this paper is to recover the linearization effect and to establish how it can be somehow controlled. For that purpose, we will consider the scaling of partition function (1) in some “mixed” asymptotic regime where, as the resolution becomes smaller, the total length of the sample is increased. Let us introduce some useful notations: We call ℓ the scale of observations (it can be the sampling scale or a multiple of it), T the integral scale and L the total sample length. N_ℓ will refer to the number of samples per integral scale and N_T the number of integral scales. We have obviously $N_\ell = T\ell^{-1}$ and $N_T = LT^{-1}$ and $N = N_\ell N_T = L/\ell$ is the total number of samples. If T is fixed, the limit $N \rightarrow +\infty$ can be conveniently controlled using an additional exponent $\chi \geq 0$ as $N_T \sim N_\ell^\chi$. Let us mention that such an exponent has been already introduced by B.B. Mandelbrot as an “embedding dimension” [12, 13] in order to discuss the concept of negative dimension and latent singularities (see below). Within this framework, the definition (1) of the partition function depends on χ and allows us

to define a new exponent $\tau_\chi(q)$ as follows:

$$Z(q, \ell, \chi) = \sum_{i=1}^{L\ell^{-1}} M[I_\ell(i)]^q \underset{\ell \rightarrow 0}{\sim} \ell^{\tau_\chi(q) - \chi} \quad (4)$$

The two “extreme” cases are : (i) the $\chi = 0$ case which corresponds to a fixed number of integral scales N_T while $\ell \rightarrow 0$ and (ii) the $\chi = +\infty$ case which corresponds to a fixed observation scale ℓ while $N_T \rightarrow +\infty$. It results that for $\chi = 0$ one recovers former definition (1) of $\tau_0(q)$ while $\tau_\infty(q) = \tau(q)$ as defined in (2). In that respect χ allows us to interpolate between quenched and annealed situations. In order to compute $\tau_\chi(q)$ one needs to study the behavior of the probability law of $Z(q, \ell, \chi)$ [14]. For that purpose, along the same line as in references [15, 16], let us study its Laplace transform: $G(s, \ell) = \mathbb{E}[e^{-e^{-qs}Z(q, \ell, \chi)}]$. Let r, p two integers and let us define the iteration $m \rightarrow m+1$, $\ell \rightarrow 2^{-p}\ell$ and $L \rightarrow 2^r L$. In other words, at each iteration step, the resolution is divided by 2^p while the number of independent integral scales is multiplied by 2^r . One thus have $N_T = 2^{rm}$ and $N_\ell = 2^{pm}$ that corresponds to $\chi = r/p$. Within this parametrization, $G(s, \ell)$ will be denoted as $G(s, m, p, r)$ and $G(s, m, 1, 0)$ will be denoted as $H(s, m)$. If M is a random cascade as defined previously, its self-similarity allows one to prove that $G(s, m, p, r)$ can be written as [17]:

$$G(s, m, p, r) = (H(s, pm))^{2^{rm}}, \quad (5)$$

where $H(s, m)$ satisfies the following recursion

$$H(s, m+1) = [H(s, m) * g(s + \ln 2)]^2 \quad (6)$$

where $g(x)$ is the pdf of ω the logarithm of cascade weights and $*$ stands for the convolution product.

It is easy to see that Eq. (6) as two uniform “stationnary” solutions $G(s, n) = 0$ and $G(s, n) = 1$, the first one being stable while the latter is (linearly) unstable. The initial condition connects the stable state $H(-\infty, 0) = 0$ to the unstable one $H(\infty, 0) = 1$ and, as shown in [15, 18, 19], this kind of non-linear equation admits traveling wave solutions $H(s, n) = H_0(s - vn)$ where v is the front velocity. This velocity can be computed by studying the linearized version of Eqs. (5) and (6) around the unstable state. After a little algebra [17], one can show that the solutions of (5) are traveling fronts that can be written, when $s \rightarrow +\infty$, as $G(s, m, p, r) = (1 - Ce^{-\gamma(x-v(\gamma)pm)})^{2^{rm}}$, provided $v(\gamma)$ satisfies the dispersion relation: $v(\gamma) = \frac{\ln(2)(r-p\tau(\gamma))}{\gamma}$ where $\tau(\gamma)$ is defined by Eq. (2). One can reproduce the same kind of analysis as in refs. [18, 20, 21] and show that a standard Aronson-Weinberger stability criterium can be used to compute the selected velocity and γ values: Let q_χ be the unique positive γ value such that $v(q_\chi) = \min_{\gamma>0} v(\gamma)$. The selected velocity $v(\gamma)$ actually corresponds to the γ value equal to the exponential decreasing rate of the initial condition provided it is greater

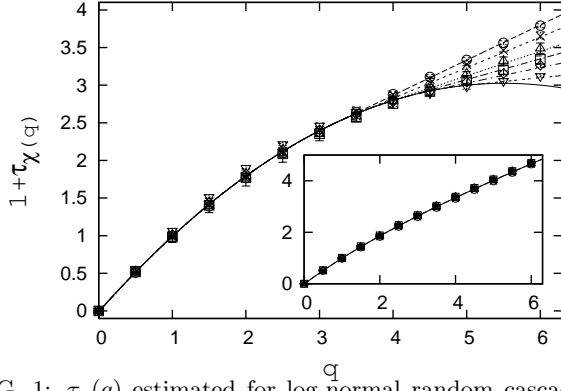


FIG. 1: $\tau_\chi(q)$ estimated for log-normal random cascade for $\chi = 0$ (\circ), 0.2 (\times), 0.5 (\triangle), 0.7 (\square), 1 (\diamond), 1.5 (∇). For each value of χ , we have chosen $T = 4096$, $\lambda^2 = 0.2$ and N_T depends on χ such that $N_T N_\ell$ is fixed. One clearly sees that, as the value of χ increases, the value of q_χ increases, the slope α_χ of $\tau_\chi(q)$ decreases and $\tau_\chi(q)$ becomes closer to the annealed spectrum $\tau(q)$ (solid line). The dashed lines represent the analytical expressions of τ_χ derived from Eqs. (8) and (9). In the inset the same estimations have been performed for a log-Poisson cascade. As expected, in that case $\tau_\chi(q)$ weakly depends on χ and rapidly converges towards $\tau(q)$ represented by the solid line (see text)

than q_χ . But the initial condition $H(s, 0)$ is precisely given by the Laplace transform of the unconditional law of $M[0, 1]^q$ and if $\mathbb{E}[M^q] = M_q < +\infty$ (such a condition is well known to be satisfied provided $\tau(q) > 0$) then $H(s, 0) \sim_{s \rightarrow \infty} 1 - M_q e^{-qs}$. Therefore, the selected velocity is simply $v = \frac{\ln(2)[r - p\tau(q)]}{q}$ if $q < q_\chi$ and $v = v(q_\chi)$ otherwise. Thanks to the fact that m is related to the resolution scale by $\ln(\ell) = -mp \ln(2)$, and using $\chi = r/p$, the velocity, measured as respect to $\ln(\ell)$, finally becomes $v(q) = \frac{\tau(q) - \chi}{q}$ if $q < q_\chi$ and $v(q) = \max_{q>0} \frac{\tau(q) - \chi}{q}$ otherwise. If one denotes $f(\alpha)$ the legendre transform of $\tau(q)$, the value q_χ for which the maximum value of $\frac{\tau(q) - \chi}{q}$ is reached, satisfies $q_\chi = f'(\alpha_\chi)$ with

$$f(\alpha_\chi) = -\chi \text{ and } \alpha_\chi = \tau'(q_\chi) = v(q_\chi) \quad (7)$$

Since positive values of $f(\alpha)$ correspond to a fractal dimension, the χ value can be seen as a kind of “negative dimension”. In order to solve the initial problem one can reproduce the analysis of [15] to show that the front velocity is directly related to the mean value of $\ln Z(q, \ell, \chi)$ as $q^{-1} \mathbb{E}[\ln Z(q, \ell, \chi)] \sim v(q) \ln(\ell)$. Since in the “moving frame”, the probability distribution of $\ln Z(q, \ell, \chi)$ converges, it results that the fluctuations of $\ln Z(q, \ell, \chi) / \ln(\ell)$ vanish. This notably implies that $q^{-1} \frac{\ln Z(q, \ell, \chi)}{\ln \ell} \rightarrow v(q)$, where the convergence is in probability. According to definition (4), this is equivalent to say that $\tau_\chi(q)$ is non random and equals to $v(q, \chi) + \chi$, i.e.:

$$\tau_\chi(q) = \tau(q) \text{ if } q < q_\chi \quad (8)$$

$$\tau_\chi(q) = q\alpha_\chi \text{ otherwise} \quad (9)$$

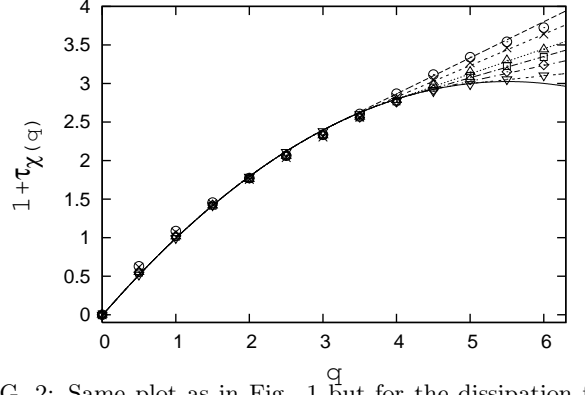


FIG. 2: Same plot as in Fig. 1 but for the dissipation field $\epsilon(x)$ estimated from a high Reynolds number jet turbulence experiment. The symbols have the same meaning as in Fig. 1. As for the log-Normal synthetic cascade one clearly observes, as χ increases, a continuous evolution of $\tau_\chi(q)$ towards the analytical parabola corresponding to a log-Normal model with $\lambda^2 = 0.2$ (solid line). Such an effect is not observed for a log-Poisson cascade.

where α_χ is defined in (7). Let us note that a rigorous proof of Eqs (8) and (9) will be provided in [14]. In the case $\chi = 0$ one recovers standard linearization effect (Eq. (3)) which has been generalized to any value of χ in the mixed asymptotic regime. As χ increases so does q_χ and $\tau_\chi(q)$ continuously converges towards $\tau(q)$. Singularities $\alpha < \alpha_0$ have been qualified by Mandelbrot as “latent” because they are only observable for large enough values of the “supersampling” exponent χ [12].

In order to illustrate our results, we have performed several numerical simulations on both continuous and discrete cascades which statistics are log-normal (LN) and log-Poisson (LP). In the log-normal case one has $\tau(q) = q(1 + \lambda^2/2) - \lambda^2 q^2/2 - 1$ and therefore by solving Eq (7) one gets $\alpha_\chi = 1 + \lambda^2/2 - \lambda\sqrt{2(1+\chi)}$ and $q_\chi = \lambda^{-1}\sqrt{2(1+\chi)}$. If Fig. 1, we have plotted, for each value of $\chi = 0, 0.2, 0.5, 0.7, 1, 1.5$, $\zeta_\chi(q) = 1 + \tau_\chi(q)$ as a function of q estimated from the scaling of partition function $Z(q, \ell, \chi)$ over a wide enough range of scales. The log-normal cascade samples are generated from discrete cascades with $T = 4096$, $\lambda^2 = 0.2$ and, for each value of χ , N_T is chosen such that, at the smallest scale, the number of sampling points is almost constant. As expected, one sees that, when the value of χ increases, q_χ increases, α_χ (the slope of $\tau_\chi(q)$ for $q > q_\chi$) decreases and one has $\tau_\chi(q) = \tau(q)$ over a wider range of q . The same kind of simulations have been performed on log-Poisson synthetic cascades for which $\tau(q) = q(1 + \lambda^2(e^\delta - 1)/\delta^2) + \lambda^2(1 - e^{q\delta}/\delta^2) - 1$. In that case, it is easy to show that since $\tau(q)$ has an asymptote when $q \rightarrow +\infty$ $\tau'(q) \geq 1 + \lambda^2(e^\delta - 1)/\delta^2$ and $f(\alpha) \geq -\lambda^2/\delta^2$. Therefore, for all $\chi \geq \lambda^2/\delta^2$, one has $\tau_\chi(q) = \tau(q)$. If one chooses the log-Poisson parameter values usually considered to model energy dissipation in turbulence [1, 22], i.e., $\lambda^2 = 0.2$ and $\delta = \ln(2/3)$, one sees that $\tau_\chi(q)$ rapidly converges towards $\tau(q)$ and no longer

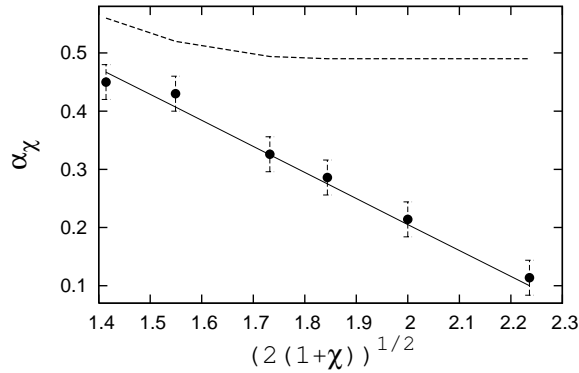


FIG. 3: Plot of α_χ , the slope of $\tau_\chi(q)$, for $q \geq q_\chi$ estimated for the turbulent dissipation field as a function of $\sqrt{1+\chi}$. The straight solid line represents the log-Normal prediction while the dashed curve is the log-Poisson prediction. Error bars have been estimated using Monte-Carlo trials of the Log-normal cascade. This representation allows us to distinguish clearly the two models of energy dissipation intermittency.

varies for $\chi \geq 0.2$. This is illustrated in the inset of Fig. 1 where all the estimated $\tau_\chi(q)$ for $\chi = 0, 0.2, 0.5, 0.7, 1, 1.5$ are close or equal to the annealed spectrum $\tau(q)$ represented by the solid line.

According to our analysis and from numerical experiments reported in Fig. 1, one sees that despite the fact that $\tau_\chi(q)$ associated with log-normal and log-Poisson are very similar for small values of q and small values of χ , the situation is very different for large values of χ and q . Since both models are traditionally used to describe the spatial fluctuations of energy dissipation in fully developed turbulence [1, 22], we naturally reproduced previous statistical analysis using experimental data of turbulence. The data have been recorded by the group of B.

Castaing in Grenoble in a low temperature gaseous Helium jet experiment which Taylor scale based Reynolds number is $R_\lambda = 929$ [23]. Assuming the validity of Taylor hypothesis and isotropy of the flow, a proxy of the dissipation field $\epsilon(x) \simeq (\partial v / \partial t)^2$ is build from the temporal longitudinal velocity signal. The overall sample is such that $N_T \simeq 2500$. The estimated values of $\tau_\chi(q)$ are reported in Fig. 2. The similarity with the results of log-normal synthetic cascades is striking: As in Fig. 1, when χ increases, the estimated $\tau_\chi(q)$ converges to the parabolic $\tau(q)$. In Fig. 3, we have plotted the values of the asymptotic slope of $\tau_\chi(q)$, α_χ , as a function of $\sqrt{1+\chi}$: one clearly sees that the data perfectly match the linear log-Normal expression (solid line) which is very different from the log-Poisson prediction (dashed line).

To summarize, we have shown in this paper, using traveling wave solutions of cascade non-linear iteration equation, that quenched and annealed averaged partitions function have different behavior for values of q larger than a critical value q_χ , analog of the glass transition temperature in spin glass systems. This difference can be controlled using some "supersampling" exponent χ which defines an asymptotic limit that mixes small scales and large number of samples regimes. By analyzing the value of the multifractal scaling exponents for various values of χ one can distinguish between different cascade models. In the context of the modelling of energy dissipation intermittency in fully developed turbulence, we have provided evidences supporting log-Normal statistics against log-Poisson statistics.

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